Maths Fundamentals (for MA11)

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Arithmetic

Whilst this document will not delve into the most basic arithmetic (tens and units, etc.), people often forget how to correctly calculate with negative numbers.

$$m - (-n) = m + n \qquad \text{(e.g., } 10 - (-5) = 15\text{)}$$

$$\begin{aligned} m \times -n &= -(m \times n) && \text{(e.g., } 10 \times -5 = -50\text{)} \\ -m \times n &= -(m \times n) && \text{(e.g., } -10 \times 5 = -50\text{)} \\ -m \times -n &= m \times n && \text{(e.g., } -10 \times -5 = 50\text{)} \end{aligned}$$

$$\begin{aligned} m \div -n &= -(m \div n) && \text{(e.g., } 10 \div -5 = -2\text{)} \\ -m \div n &= -(m \div n) && \text{(e.g., } -10 \div 5 = -2\text{)} \\ -m \div -n &= m \div n && \text{(e.g., } -10 \div -5 = 2\text{)} \end{aligned}$$

And finally, let's not forget the golden rule when multiplying numbers by zero: $n \times 0 = 0$.

Fractions

For the addition and subtraction of fractions, always remember that the denominator (bottom) must be the same for each fraction involved. Then, simply apply the following (for subtraction, simply substitute the pluses for minuses):

$$\tfrac{n}{p} + \tfrac{m}{p} = \tfrac{n+m}{p} \qquad \text{(e.g., } \tfrac{2}{7} + \tfrac{3}{7} = \tfrac{2+3}{7} = \tfrac{5}{7}\text{)}$$

Unfortunately, you will not always be given fractions with a common denominator. It will therefore be necessary to perform some extra calculations to get the expression into the required format, before addition/subtraction can take place.

$$\tfrac{1}{3} + \tfrac{2}{9}$$

In the above example, the lowest common denominator is 9. The fraction on the right can be left as we found it, as its denominator is already 9. To convert the left-hand fraction, we simply multiply the top and bottom numbers by some constant that will result in the new denominator being 9. In this case, we must multiply by 3. The addition can then be carried out as normal:

$$\frac{1}{3} + \frac{2}{9} = \frac{1\times3}{3\times3} + \frac{2}{9} = \frac{5}{9}$$

The next eventuality is when neither dominator shares a common factor. The trick then is to multiply the two denominators together, with the result being a number where the two denominators are obviously factors. It is then a matter of multiplying each fraction by the required constant. It's easier to see with an example or two:

$$\frac{1}{3} + \frac{1}{2} = \frac{1\times2}{3\times2} + \frac{1\times3}{2\times3} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$

$$\frac{2}{5} + \frac{4}{9} = \frac{2\times9}{5\times9} + \frac{4\times5}{9\times5} = \frac{18}{45} + \frac{20}{45} = \frac{38}{45}$$

Adding fractions to whole numbers is not too taxing. Remember, that a whole number, such as 5 is equivalent to the fraction $\frac{5}{1}$. Then it is a matter of following the above instructions on finding a common denominator.

$$\frac{n}{m} + p = \frac{n+mp}{m} \qquad \text{(e.g., } \frac{1}{4} + 5 = \frac{1}{4} + \frac{5}{1} = \frac{1}{4} + \frac{5\times4}{1\times4} = \frac{1}{4} + \frac{20}{4} = \frac{21}{4}\text{)}$$

Multiplying fractions is very simple. Just take it a row at a time, and then treat is as though it is a straight-forward multiplication. The general case:

$$\frac{m}{n} \times \frac{p}{q} = \frac{m\times p}{n\times q} \qquad \text{(e.g., } \frac{1}{10} \times \frac{2}{3} = \frac{1\times2}{10\times3} = \frac{2}{30} = \frac{1}{15}\text{)}$$

Always remember to simplify the resulting fraction if possible.

Dividing fractions is almost as easy. The method comes down to the fact that dividing by a particular number is equivalent to multiplying by its reciprocal[1].

$\frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \frac{q}{p} \qquad \text{(e.g., } \frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{2}{4} = \frac{1}{2}\text{)}$

As you should have noticed, the fraction which you are diving by is flipped upside-down, and division operator is replaced by the multiplication sign.

Simplifying & Re-arranging equations

This section will only outline a handful of useful techniques that will come in useful to manipulate equations.

Cancelling out terms is very useful for simplifying equations. As its name suggests, where possible, terms can be legally removed completely from an expression. Opportunity for cancelling out occurs whenever fractions are present. For example:

$$\frac{2x}{x} = \frac{2}{1} \times \frac{x}{x} = \frac{2}{1} \times 1 = 2$$

[1] Feel free to quickly look up what reciprocal means!

Refresh your mind on how to multiply fractions from the previous section if you didn't follow the example. What the example tries to illustrate is that where a number appears in both the numerator and denominator, then these cancel each other out, since a number divided by itself is always 1. More examples:

$$\begin{aligned} \frac{3x^2}{x} &= 3x \\ \frac{n(n+1)(n-1)}{(n+1)} &= n(n-1) \\ \frac{4xyz}{2xz} &= 2y \end{aligned}$$

When the numerator contains more than one term, then you cannot cancel from the denominator unless it is a factor for each term. E.g.,

$$\tfrac{4x^2+2x}{x} = \tfrac{4x^2}{x} + \tfrac{2x}{x} = 4x + 2$$

Manipulating equations in order to state an equation in an alternative way is frequently relied upon. Typically, the task involves isolating a particular term of interest. For example, solve for x in the following equations:

$$\begin{aligned} 3x + 2 =& 26 && \text{(subtract 2 from both sides)} \\ 3x =& 24 && \text{(divide both sides by 3)} \\ x =& \ 8 \end{aligned}$$

$$\begin{aligned} \frac{4x+y}{2} =& \quad z && \text{(multiply by 2)} \\ 4x + y =& \quad 2z && \text{(subtract y)} \\ 4x =& \ 2z - y && \text{(divide by 4)} \\ x =& \frac{2z-y}{4} \end{aligned}$$

Factorising

To quickly recap: a *factor* is a number that can divide another number without leaving any remainder (e.g., 2 goes in to 10, five times exactly.) *Common factors* are numbers that can divide into two or more numbers (e.g., 1, 2, 4 are common factors of 8 and 12.)

The task of factorising is to extract common factors from terms within an expression, such that when the factors are multiplied by the remaining expression, you get the original expression back.

$$\begin{aligned} 10xy^2 + 15x^2y - 5xy &= 5(2xy^2 + 3x^2y - xy) \\ &= 5x(2y^2 + 3xy - y) \\ &= 5xy(2y + 3x - 1) \end{aligned}$$

A quadratic equation comes in the form of ax^2+bx+c. When factorised, you get a result in the form of $(x+m)(x+n)$[2]. If you multiply out this factorised form, you get: $x^2+(m+n)x+mn$. Therefore, the skill to factorising quadratics is to find two numbers, m and n, such that when added together, give the value of b, and when multiplied together, give c.

Factorise x^2+5x+6.

Factors of 6		+	×
1	6	7	6
2	**3**	**5**	**6**

Therefore, the factors are $(x+2)(x+3)$.

Of course, b or c could be negative, which requires a little extra effort:

Factorise $x^2-7x+12$. Since b is negative, and c is positive, then it must be the case that m and n are either both positive or both negative.

Factors of 12		+	×
1	12	13	12
-1	-12	-13	12
2	6	8	12
-2	-6	-8	12
3	4	7	12
-3	**-4**	**-7**	**12**

Therefore the factors are $(x-3)(x-4)$.

Factorise $x^2+3x-10$. This time, since c is negative, then it means that either m or n is negative, but not both.

Factors of 10		+	×
1	-10	-9	-10
-1	10	9	-10
2	-5	-3	-10
-2	**5**	**3**	**-10**

Therefore the factors are $(x-2)(x+5)$.

Powers & Indices

Take the expression, a^m. This reads as "a to the power of m", where a is the base, and m is the index. Many people will be familiar with expressions where $m=2$, since terms such as x^2 are very common in algebra. However, a little more knowledge about indices will be very beneficial to this course.

$$\begin{aligned} a^0 &= 1 \\ a^1 &= a \\ a^2 &= a \times a \\ a^3 &= a \times a \times a \\ etc\ldots \end{aligned}$$

[2]This is true when $a=1$. An exercise for the reader is to see if you can figure out the procedure for factorising when $a>1$.

Multiplying:

$$a^m \times a^n = a^{(m+n)}$$

Powers of powers:

$$(a^m)^n = a^{(m \times n)}$$

Dividing:

$$a^m \div a^n = a^{(m-n)}$$

Fractions:

$$\left(\frac{x}{y}\right)^m = \left(\frac{x^m}{y^m}\right)$$

Negative indices:

$$a^{-m} = \frac{1}{a^m}$$

Fractional indices:

$$a^{\frac{x}{y}} = \sqrt[y]{a^x} \qquad \text{(e.g., } a^{\frac{1}{2}} \text{ is equivalent to standard square root)}$$