More Examples of Logical Equivalence

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Logical equivalence can be tricky to begin with until you get enough practice. There are a couple of examples that demonstrate how the laws are used to show an equivalence. There are also a few questions for you to attempt. The answers are included on the final page of this worhsheet, however, you will not learn much from this exercise if you do not attempt them first.

Examples

1. Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{(De Morgan's law)} \\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{(De Morgan's law)} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{(Double negation law)} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{(Distributive law)} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{(Contradiction law)} \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{(Communitive law)} \\
&\equiv \neg p \wedge \neg q && \text{(Identity law)}
\end{aligned}$$

2. Given the Law of Implication ($p \to q \equiv \neg p \vee q$), show that $(p \wedge q) \to (p \vee q)$ is a tautology (i.e., it is equivalent to true.)

$$\begin{aligned}
(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{(Law of implication)} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{(De Morgan's law)} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{(Associative law)} \\
&\equiv T \vee T && \text{(Excluded middle law)} \\
&\equiv T && \text{(Domination law)}
\end{aligned}$$

Questions

N.B. Bear in mind that in order to reach the end target, you may occasionally need to make non-obvious steps, which initially could make things more complicated. Perserverance and practice will hopefully guide you as to when you need to make such choices.

1. Show that $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$.
2. Show that $((p \wedge q) \wedge r) \wedge s \equiv ((s \wedge r) \wedge q) \wedge p$.
3. Show that $p \vee (p \wedge q) \equiv p$.
4. Show that $p \vee q \equiv \neg(\neg p \wedge \neg q)$.

Answers

1.

$$\begin{aligned}\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) &\equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q) && \text{(De Morgan's law)}\\ &\equiv \neg p \wedge (q \vee \neg q) && \text{(Distributive law)}\\ &\equiv \neg p \wedge T && \text{(Excluded middle law)}\\ &\equiv \neg p && \text{(Identity law)}\end{aligned}$$

2.

$$\begin{aligned}((p \wedge q) \wedge r) \wedge s &\equiv s \wedge ((p \wedge q) \wedge r) && \text{(Commutive law)}\\ &\equiv s \wedge (r \wedge (p \wedge q)) && \text{(Commutive law)}\\ &\equiv s \wedge (r \wedge (q \wedge p)) && \text{(Commutive law)}\\ &\equiv (s \wedge r) \wedge (q \wedge p) && \text{(Associative law)}\\ &\equiv ((s \wedge r) \wedge q) \wedge p && \text{(Associative law)}\end{aligned}$$

3.

$$\begin{aligned}p \vee (p \wedge q) &\equiv (p \wedge T) \vee (p \wedge q) && \text{(Identity law)}\\ &\equiv (p \wedge (q \vee \neg q)) \vee (p \wedge q) && \text{(Ecluded middle law)}\\ &\equiv (p \wedge q) \vee (p \wedge \neg q) \vee (p \wedge q) && \text{(Distributive law)}\\ &\equiv (p \wedge q) \vee (p \wedge \neg q) && \text{(Idempotent law)}\\ &\equiv p \wedge (q \vee \neg q) && \text{(Idempotent law)}\\ &\equiv p \wedge T && \text{(Excluded middle law)}\\ &\equiv p && \text{(Identity law)}\end{aligned}$$

4.

$$\begin{aligned}p \vee q &\equiv \neg\neg p \vee q && \text{(Double negative law)}\\ &\equiv \neg\neg p \vee \neg\neg q && \text{(Double negative law)}\\ &\equiv \neg(\neg p \vee \neg q) && \text{(De Morgan's law)}\end{aligned}$$