Combinatorics Part II

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Before going further, it is first important that you are familiar with factorials. The notation is a number (or symbol) followed by an exclamation mark, e.g., 5! The general case:

$$n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$$

So you are multiplying each natural number between n and 1 to calculate the factorial. See your notes for further information.

Permutations

These relate to specific types of selections where order is important, but no repetitions are allowed.

Examples:

1. Given the letters A, B, C, and D. How many permutations of these four letters are there?

   This should not be too taxing, as it is easily solved using the multiplication principle.

   Pick first letter (4 choices)
   Pick second letter (3 choices)
   Pick third letter (2 choices)
   Pick fourth letter (1 choice)
   $\Rightarrow 4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

2. Given 6 people standing in a line, how many possible ways can they be arranged?

   Just like the example above, but with 6 objects instead of 4.
   $\Rightarrow 6! = 720$

3. The same 6 people are about to sit down at a round table for a meal. How many different ways could they sit themselves?

   Initially, you may think that the answer would be the same as the answer for the previous question. However, as you can see from the picture of a potential table layout, rotations will be considered identical, which means ABCDE = BCDEA, which was not the case when standing in a line.

   To solve, position A arbitrarily (1 choice). To seat the remaining 5 people, we can order them as usual. Therefore, there are 5! permutations.

It is not always the case that the total number of permutations is sought. Alternatively, you may only be interested in a subset of the objects in question. For example, from the 6 people as mentioned in previous examples, how many ways can they be paired. This is known as a k-permutation[1].

To select k objects from a set of n objects is denoted as $P(n, k)$. To calculate:

$$P(n,k) = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

You may be pleased to know that this function is found on most calculators (look for nPr), which is useful for checking that your working out is correct.

More examples:

4. Given the letters A, B, C, D, and E.

5. How many possible 5 letter permutations are there (assuming no repeats)?

   This should now be very obvious to you.
   $\Rightarrow P(5, 5) = 5!$

6. How many of the above strings contain the substring 'ACE'?

   If ACE must stay together in that order, then we can use a simple trick of combining them and treating them as a single entity. With the remaining 2 letters, there are only 3 objects to consider. Therefore, there are 3! permutations that contain the substring ACE.

7. How many strings contain neither of the substrings AB, CD?

   The approach here is to find the number of strings that do contains either AB or CD and then subtract from the total number of string possible.

   To find the number of strings containing AB, use the same principle as in part b). Treat AB as a single entity, and there are 3 other letters. Thus, 4! strings with AB.

   That is the same result for strings with CD. Therefore, the total number of strings containing either AB or CD $= 4! + 4! = 2 \cdot 4!$

   Now for the tricky part! Within the set of permutations that contain AB, there are some that also contain CD. It is therefore also true that for the set of strings containing CD, there are some that also contain AB. This means that the above result of $4! + 4!$ actually contains some duplicates when added together. To reach the final answer, we must remove this duplication.

   This is actually fairly trivial to work out once you know what you are looking for. The duplication was caused by counting the number of strings that contain AB and CD twice. The number of strings that contain AB and CD is 3!. Therefore, if we subtract this amount from the $2 \cdot 4!$ strings that contain

[1] Is also known as an r-permutation, which is probably more common, especially in the maths literature. However, it is only a name, and so I shall stick with the way your notes present the material.

either AB or CD, then we will have removed the duplicates.

Finally, we already know the total number of permutations from part *a)*. So, the answer to the original question is $5! - ((2 \cdot 4!) - 3!) = 78$

8. How many strings contain neither of the substrings AB, BE?
Even though it looks like an identical question to part *c)*, it is subtly different. Lets see how it is different.

   Go through the steps as the previous part. The difference comes in the form that there is an extra condition, i.e., a string can only contain AB and BE if and only if it contains ABE. Part *b)* gave us the answer for the number of permutations for substrings with three letters, 3!.
$\Rightarrow 5! - (2 \cdot 4! - 3!) = 66$.

Combinations

These are similar to permutations, however, order is not important, but still no repetitions are allowed.

The notation for k combinations is $C(n, k)$ where n is the number of distinct elements.

$$C(n, k) = \frac{P(n, k)}{n!}$$

Examples:

1. In the main draw of the National Lottery, the player can select 6 numbers from a possible 49. How many possible combinations are there?

$$\Rightarrow C(49, 6) = \frac{P(49, 6)}{6!} = 13,983,816$$

   Now you should see where the 14million-to-1 odds come from!

2. In a game of poker, you are dealt a hand of 5 cards from a pack of 52.

   (a) How many possible hands are there?

   Again, order is still not important because A2345 is the same hand as 5432A.

$$\Rightarrow C(52, 5) = \frac{P(52, 5)}{5!} = 2,598,960$$

   (b) How many hands contain cards of exactly one suit?

   There are only 13 cards in each suit. So the possible number of hands is $C(13, 5)$ for a suit. And since there are four suits available, the total number of hands is $4 \cdot C(13, 5)$.

   (c) How many hands contain cards of exactly two suits?

   Imagine the suits in question are Spades and Hearts (the calculations are the same for any two, however, its easier to be able to refer to suits with a name). There are 26 cards between the two, which means there are $C(26,5)$ ways to select 5 cards. However, don't forget, within that selection, there will be $C(13,5)$ that only contain spades, and the same amount that only contain hearts. This leaves us with $C(26,5)-2\cdot C(13,5)$ possible hands that contain Spades and Hearts. We must now generalise for any combination of two suits. The number of ways to choose two suits from four is $C(4,2)$.

$$\Rightarrow C(4,2)\cdot(C(26,5)-2\cdot C(13,5)) \text{ possible hands.}$$

3. There are 10 members in a tennis club. They are to be paired up to make 5 doubles for a forthcoming tournament. How many ways can this be done?

   Start off by treating the grouping each pair as a separate step:

   1. Choose first pair, $C(10,2)=45$.
   2. Choose second pair, $C(8,2)=28$.
   3. Choose third pair, $C(6,2)=15$.
   4. Choose fourth pair, $C(4,2)=6$.
   5. Choose fifth pair, $C(2,2)=1$.

   $\Rightarrow 45\cdot 28\cdot 15\cdot 6\cdot 1 = 113,400$.

   However, order of the 5 doubles is not important, so we need to divide by the number of permutations that the groups can be arranged, 5!.

$$\Rightarrow \frac{113400}{5!} = 945 \text{ possible groups.}$$

Permutations with Unlimited Repetition

This is quite possibly the easiest type of counting method that you have to learn (in my opinion!) If you wish to select k objects, where repetition is allowed, from a total set of n objects, then the resulting number of choices is n^k.

Examples:

1. A roulette wheel has the numbers 0 to 36 (inclusive) and for each turn, a ball will land on only one of those numbers. After 5 turns at the wheel, how many possible sequences are there?

   Since each turn is independent from any other, there are always 37 possibilities for each turn. Therefore, after 5 turns,
   $\Rightarrow 37\cdot 37\cdot 37\cdot 37\cdot 37 = 35^5$.

2. A die a rolled 3 times. How many possible permutations are there?

   This is very simple. There are 6 choices per roll.
   $\Rightarrow 6\cdot 6\cdot 6 = 6^3$.

I could go on and produce a massive list of examples for this type of permutation. However, all questions of this type will be in a similar style, just the preamble that surrounds the key attributes is different, e.g., dice, cards, bags of sweets etc.

Combinations with Unlimited Repetition

When the repetition is allowed, working out combinations changes slightly. To select k combinations from n objects:

$$C(k+n-1,k) = \frac{(k+n-1)!}{k!(n-1)!}$$

Section 2.2.4 in your notes goes into detail as to how the above formula comes into existence.

Summary

You would be well advised to have the following table at hand when doing your combinatorics coursework, and to try and memorise for your exam.

		Selections	
		Ordered	*Unordered*
Repetition	*None*	$P(n,k)$	$C(n,k)$
	Unlimited	n^k	$C(k+n-1,k)$