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Systems of Linear Equations

$2x + y = 2$ (1)
$3x - y = 8$ (2)

(1) and (2) are examples of linear equations. In this instance, there are two equations and two unknowns.

To solve:

From (1): $y = 2 - 2x$
Substitute in to (2): $3x - (2 - 2x) = 8$
Expand: $3x - 2 + 2x = 8$
Simplify: $5x - 2 = 8$
Isolate x: $\Rightarrow 5x = 10 \qquad \therefore \quad x = \frac{10}{5} = 2$
Substitute x into (1): $2(2) + y = 2$
$\Rightarrow y = 2 - 4 = -2$

There are of course a number of different ways of solving such equations depending on whether you choose to isolate *x* first or *y*, and from which equation etc. So, don't be alarmed if other peoples workings are not the same as yours, the values of the unknown variables will be the same regardless of the path chosen.

Echelon Form

Be sure you are familiar with *augmented matrices* before continuing. (see section 1.1 in notes)

Definition: If the *leading entry* of a row is the first non-zero entry in that row, then a matrix is in *echelon form* if it has the following three properties:

- i) The leading entry of each row is to the right of the leading entry of the preceding row.
- ii) All entries directly below the leading entry of each row are zeros.
- iii) All rows that consist entirely of zeros are below any rows that contain non-zero entries.

Examples of matrices in echelon form:

$$\left[\begin{array}{ccc|c} 4 & 3 & 5 & 2 \\ 0 & -2 & 3 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Examples of matrices <u>not</u> in echelon form:

$$\left[\begin{array}{ccc|c} 4 & 3 & 5 & 2 \\ 0 & -2 & 3 & 7 \\ -2 & 0 & 1 & -3 \end{array}\right], \left[\begin{array}{ccc|c} 2 & 1 & -3 & 8 \\ 0 & 0 & 3 & 6 \\ 0 & -1 & 2 & 5 \end{array}\right], \left[\begin{array}{cc|c} 1 & -3 & -7 \\ 2 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Augmented echelon matrices are very useful for solving systems of linear equations as we will see shortly.

Let's have system that already satisfies echelon form when it's converted into an augmented matrix.

$$\begin{array}{ccccccc} 2x & + & y & - & 3z & = & -6 \\ & - & 2y & + & z & = & 5 \\ & & & & 4z & = & 12 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 2 & 1 & -3 & -6 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 4 & 12 \end{array}\right] \quad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

When the system is in echelon form, it is possible to solve using back substitution. It is trivial to find the single unknown in (3), which can then be substituted in to (2) which would leave only one unknown and so on.

To solve the above example:

From (3): $4z = 12 \quad \therefore \quad z = \frac{12}{4} = 3$

Substitute *z* in to (2): $-2y + (3) = 5$

Find *y*: $y = \frac{5-3}{-2} = \frac{2}{-2} = -1$

Substitute *y* and *z* into (1): $2x + (-1) - 3(3) = -6$

Simplify: $2x - 10 = -6 \quad \therefore \quad x = \frac{10-6}{2} = \frac{4}{2} = 2$

Gauss Elimination Method

This is a very efficient way of solving linear systems containing many unknown variables. It is needed because linear systems are rarely given conveniently in echelon form.

Example:

$$\begin{array}{ccccccc} 2x & + & y & - & z & = & -1 \\ x & - & 2y & + & z & = & 5 \\ 3x & - & y & - & 2z & = & 0 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 \\ 3 & -1 & -2 & 0 \end{array}\right] \quad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

To begin, you should try to eliminate the values under the leading entry of (1). To turn the leading entry of (2) into 0, think for a multiple of 2, that when added to 1, equals 0. (i.e., $2m + 1 = 0 \quad \therefore \quad m = \frac{-1}{2}$)

So, multiply (1) by $\frac{-1}{2}$ and add to (2):

$$(1)\times\tfrac{-1}{2} = \left[\tfrac{-2}{2} \quad \tfrac{-1}{2} \quad \tfrac{1}{2} \;\middle|\; \tfrac{1}{2}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & \frac{-5}{2} & \frac{3}{2} & \frac{11}{2} \\ 3 & -1 & -2 & 0 \end{array}\right]$$

To eliminate the 3 in (3), multiply (1) by $\frac{-3}{2}$ and add to (3):

$$(1)\times\tfrac{-3}{2} = \left[-3 \quad \tfrac{-3}{2} \quad \tfrac{3}{2} \;\middle|\; \tfrac{3}{2}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & \frac{-5}{2} & \frac{3}{2} & \frac{11}{2} \\ 0 & \frac{-5}{2} & \frac{-1}{2} & \frac{3}{2} \end{array}\right]$$

Almost there! Last step for echelon form is to eliminate the leading entry of (3). This time, we will use a multiple from (2) in order to ensure the value in the first column remains zero too.

So, multiply (2) by -1 and add to (3):

$$(2)\times -1 = \left[0 \quad \tfrac{5}{2} \quad \tfrac{-3}{2} \;\middle|\; \tfrac{-11}{2}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ 0 & \frac{-5}{2} & \frac{3}{2} & \frac{11}{2} \\ 0 & 0 & -2 & -4 \end{array}\right]$$

Finally in echelon form!

This system can finally be solved.

From (3): $-2z = -4 \qquad \therefore \quad z = \dfrac{-4}{-2} = 2$

Substitute *z* into (2): $\dfrac{-5}{2}y + \dfrac{3}{2}(2) = \dfrac{11}{2}$

Simplify: $\Rightarrow \quad \dfrac{-5}{2}y + 3 = \dfrac{11}{2}$

$$\Rightarrow \quad \frac{-5}{2}y = \frac{11}{2} - 3 = \frac{5}{2}$$

$$\therefore \quad y = \frac{5}{2} \div \frac{-5}{2} = \frac{5}{2} \times \frac{2}{-5} = \frac{10}{-10} = -1$$

Substitute *y* and *z* into (1): $2x + (-1) - (2) = -1$

Simplify: $2x - 3 = -1$

$$\Rightarrow \quad 2x = -1 + 3 = 2 \quad \therefore \quad x = 1$$

More Linear Equations

Examples so far have only shown where there is only one solution to a system. It should be obvious that there are occasions where there are no solutions. There are also conditions where there are an infinite number.

Example with no solutions: (from Truss[1])

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 1 \\ 2x & - & y & + & 3z & = & 2 \\ 3x & & & + & 5z & = & 4 \end{array} \quad\Rightarrow\quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & -1 & 3 & 2 \\ 3 & 0 & 5 & 4 \end{array}\right] \quad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

The above system is fairly obviously inconsistent, if you notice that (3) is simply the sum of (1) and (2), which you would expect to equal 3 but does not. However, you will not always be so lucky to be given a system with no solutions that is as obvious as this. You will normally need to get into echelon form before you can determine its solvability. Let's see what happens.

Multiply (1) by -2 and add to (2):

$$(1) \times -2 = [-2 \quad -2 \quad -4 \mid -2]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -3 & -1 & 0 \\ 3 & 0 & 5 & 4 \end{array}\right]$$

Multiply (1) by -3 and add to (3):

$$(1) \times -3 = [-3 \quad -3 \quad -6 \mid -3]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & -3 & -1 & 1 \end{array}\right]$$

Multiply (2) by -1 and add to (3):

$$(2) \times -1 = [0 \quad 3 \quad 1 \mid 0]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \textit{Now in echelon form.}$$

What should now be immediately apparent is that (3) is not valid (i.e., $0x + 0y + 0z = 0 \neq 1$). No choice of *x, y* and *z* will make that statement valid, therefore, this system has **no solution**.

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[1] Truss, J. *Discrete Mathematics for Computer Scientists,* (2nd Edition), Addison-Wesley, 1999.

An example with infinite number of solutions:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 4z & = & 3 \\ 4x & - & 2y & - & 6z & = & 2 \\ 2x & - & y & - & 3z & = & 1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 4 & -2 & -6 & 2 \\ 2 & -1 & -3 & 1 \end{array}\right] \quad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

I shall skip the working out and simply give the above augmented matrix in echelon form. For those you fancy some more practice with the Gauss elimination method, feel free to see if you can get the correct matrix below.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & -10 & -22 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \textit{In echelon form.}$$

This time, (3) is a valid equation (the answer is 0, as you would expect), however, it is clearly redundant since it provides no information so it will be discarded.

The problem now arises that there are fewer equations than there are unknowns, which means there are **infinitely many solutions**, as we shall see.

From (2): $z = \dfrac{10-10y}{22} = \dfrac{5}{11} - \dfrac{5}{11}y$

Substitute z into (1): $x + 2y + 4(\dfrac{5}{11} - \dfrac{5}{11}y) = 3$

Simplify: $x + \dfrac{2}{11}y + \dfrac{20}{11} = 3$

Solve for x: $x = \left(3 - \dfrac{20}{11}\right) - \dfrac{2}{11}y$

$\Rightarrow \quad x = \dfrac{13}{11} - \dfrac{2}{11}y$

We now have x and z in terms of y.

Let $y = 0$

$$\Rightarrow \quad x = \frac{13}{11} - \frac{2}{11}(0) = \frac{13}{11}$$

$$\Rightarrow \quad z = \frac{5}{11} - \frac{5}{11}(0) = \frac{5}{11}$$

Let $y = 1$

$$\Rightarrow \quad x = \frac{13}{11} - \frac{2}{11}(1) = \frac{13}{11} - \frac{2}{11} = 1$$

$$\Rightarrow \quad z = \frac{5}{11} - \frac{5}{11}(1) = 0$$

And so on…

Inverse Matrices

Given a square matrix, A. Let B also be a square matrix of the same size as A. B is the inverse of A if:

$$AB = BA = I \qquad \text{(where } I \text{ is the identity matrix)}$$

Your notes show how it is possible to find the inverse of A (written as A^{-1}) using two sets of simultaneous equations. However, we will use a more efficient method, which is essentially the same as the elimination method used to solve linear systems. It is as follows:

Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ (Taken from Truss)

Store in an augmented matrix where the answer is the identity:

$$\left(\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right) \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

The method is to use basic row manipulation to transform the left-hand side into the identity. Once complete, the result on the right-hand side will be A^{-1}. Let's see how it works.

Firstly, you need to eliminate the '2' from (1) and the '4' from (2) into zeros. So, multiply (1) by $\frac{-4}{3}$ and add to (2):

$$(1)\times\tfrac{-4}{3} = \left(-4 \quad \tfrac{-8}{3} \;\middle|\; \tfrac{-4}{3} \quad 0\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{-4}{3} & 1 \end{array}\right)$$

Multiply (2) by -6 and add to (1):

$$(2)\times -6 = \left(0 \quad -2 \;\middle|\; 8 \quad -6\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 3 & 0 & 9 & -6 \\ 0 & \frac{1}{3} & \frac{-4}{3} & 1 \end{array}\right)$$

Next, all we need to do is make the '3' in (1) and the $\frac{1}{3}$ in (2) equal to 1. So, simply multiply each row by the required value to achieve this. Therefore, multiply (1) by $\frac{1}{3}$ and (2) by 3:

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{array}\right)$$

Therefore, $A^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$

Always check that $AA^{-1} = I$.