Probabily Theory — Part I

16th March 2004

Before you can proceed, you must be sure you understand the basics. These are covered at the beginning of section 3 in your notes. Terms that you need to remember are: *experiment*; *sample space*; *event*; *union* ($A \cup B$); *intersection* ($A \cap B$); *compliment* ($\bar{A}$); *mutually exclusive* ($A \cap B = \emptyset$).

Remember, all probabilities result in a number between 0 and 1 (meaning impossible and certainty respectively). It will be more than likely that most answers will get something in between. To make life easier for yourselves (and the markers!) try and keep your workings and answers as fraction rather than decimal form.

Finite Probability Spaces

Examples:

1. A die is loaded so that the numbers 2, 4 and 6 are equally likely to appear. 1, 3, and 5 are also equally likely to appear, but 1 is three times more likely as 2 to appear.

   (a) One die is rolled. Assign the probabilitues to the outcomes that accurately model the likelihood of the various numbers to appear.

   Odd numbers are 3 times more likely than even numbers. Therefore $P(2) = P(4) = P(6) = \frac{1}{12}$. $P(1) = P(3) = P(5) = \frac{3}{12}$.

   (b) One die is rolled. What is the probability of getting a 5?

   We can find this by simply looking at $P(5)$ from the previous part. Therefore, the chances of getting a 5 is $\frac{3}{12}$.

   (c) One die is rolled. What is the probability of getting an even number?

   Again, using the probability space from part *a)* we can simply add the probabilities for each of the even numbers. $P(2)+P(4)+P(6) = \frac{3}{12}$.

(d) One die is rolled. What is the probability of not getting a 5?

Part *b)* gives us the probability of getting just a 5. So, the answer for this is merely the complement of that event. Therefore, $1 - \frac{3}{12} = \frac{9}{12} = \frac{3}{4}$.

(e) Two dice are rolled. What is the probability of getting doubles?

Let's first look at the probability of just getting two 2's. $P(2,2) = P(2) \cdot P(2) = (\frac{1}{12})^2$. It is the same chances for all even number doubles. The chances of getting an odd number double, e.g., two 1's, $P(1,1) = P(1) \cdot P(1) = (\frac{3}{12})^2$ and again, it's the same for all odd doubles. Therefore, the total probability for any double is: $3(\frac{1}{12})^2 + 3(\frac{3}{12})^2$.

(f) Two dice are rolled. What is the probability of getting a sum of 7?

There is not really an easy way to discover all the combinations that result in 7 other than to simply list them yourself. $P(1,6)+P(2,5)+P(3,4)+P(4,3)+P(5,2)+P(6,1) = 3(\frac{3}{12} \cdot \frac{1}{12})+3(\frac{1}{12} \cdot \frac{3}{12}) = 6(\frac{3}{12} \cdot \frac{1}{12})$.

Equiprobable Spaces

If all the sample points within a given finite probability space are equal to each other, then it is known as an *equiprobable space*. An example would be a fair die, where each number is equally possible, i.e., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$.

At this point, you would normally be presented with yet another barrage of standard coin flipping or dice throwing examples. However, I shall now give you a couple of large, more applied examples, of which I hope you can relate to a little easier.

1. The UK National Lottery (or 'Lotto' as it is now known) is a game where the player picks 6 balls from a possible 49. At the draw, 6 numbers are selected at random from the same 49, and a bonus ball too.

(a) What are the odds of getting the jackpot? (The player picks the same 6 balls as the 6 chosen at the draw.)

You should have seen this example before. Using standard counting methods, to choose 6 numbers from 49 is $C(49,6)$. Therefore, the odds are $\frac{1}{C(49,1)}$ or 1 in $13,983,816$.

(b) What are the odds of getting just 5 numbers and the bonus ball?

You've matched 5 of the winning 6 balls. So, you need to work out the number of ways of not getting the 6^{th} winning ball. There are 43 numbers remaining, and to choose one of them, is simply $C(43, 1) = 43$. Multiply that with the number of ways of getting 5 numbers from the winning 6: $C(6, 5) \cdot C(43, 1) = 6 \cdot 43 = 258$.

So, there are 258 combinations that match 5 numbers. The chances of your 6^{th} number matching the bonus ball is $\frac{1}{43}$. Therefore, the number of combinations of getting 5 numbers and the bonus is $\frac{1}{43} \cdot 258 = 6$. The final odds are:

$$\frac{6}{13,983,816} = 1 \text{ in } 2,330,636$$

(c) What are the odds of getting just 5 numbers but not the bonus ball?

The workings from the previous part told us that there are 258 ways to select 5 from the winning 6 numbers. Also, we saw that of those 258, there were 6 combinations that included the bonus ball. Therefore, to get 5 numbers only $258 = 6 = 252$.

$$\frac{252}{13,983,816} = 1 \text{ in } 55,491 \text{ (when rounded).}$$

(d) What are the odds of getting just 4 balls?

Now that we can forget about bonus numbers, the problem becomes much easier. Firstly, like with the previous parts, we need to find how many ways there are of choosing 4 numbers from the winning 6. Then, we need to work out how many ways to choose the remaining two numbers. $C(6, 4) \cdot C(43, 2) = 15 \cdot 903 = 13,545$ possible winning 4 number combinations.

$$\frac{13,545}{13,983,816} = 1 \text{ in } 1,032 \text{ (when rounded).}$$

(e) What are the odds of getting just 3 balls?

Just follow the same steps as in the previous part, except slightly changing the numbers as required. $C(6, 3) \cdot C(43, 3) = 20 \cdot 12,341 = 246,820$.

$$\frac{246,820}{13,983,816} = 1 \text{ in } 57 \text{ (when rounded).}$$

2. The birthday problem: if there are n people in a room, what is the probability that at least two people will share the same birthday? (We assume that each day in the year is as equally likely as a birthday, and we will ignore leap years.)

The best approach is to first find the complement of the above event, i.e., that no two people share the same birthday.

Let there only be two people in the group ($n = 2$). This means there are 365 possible birthdays for the first person. And for the second person not to have the same birthday, there are 364. This must then be divided by the total sample space, that is 365×365 possible combinations.

$$P(\text{'no shared birthdays'}) = \frac{365 \cdot 364}{365^2}$$

When $n = 3$:

$$P(\text{'no shared birthdays'}) = \frac{365 \cdot 364 \cdot 363}{365^3}$$

When $n = 4$:

$$P(\text{'no shared birthdays'}) = \frac{365 \cdot 364 \cdot 363 \cdot 362}{365^4}$$

A pattern can quickly be spotted, which gives a general formula for all n:

$$P(\text{'no shared birthdays'}) = \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}$$

Now that the complement has been found, we simply subtract that probability from 1 and we have the final result:

$$P(\text{'shared birthdays'}) = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}$$

Now let us investigate some values of *n:* (Note: all values have been rounded to 4dp)

n	Prob	n	Prob
5	0.0271	23	0.5073
10	0.1169	30	0.7063
20	0.4114	40	0.8912
22	0.4757	50	0.9704

Are you surprised that in a room with 23 people, the probability is over $\frac{1}{2}$?

3. In a game of poker, a player is dealt 5 cards from a pack of 52. What are the probabilities for each of the possible hands that can be played? (Note: order in which a player is dealt the cards is not relevant.)

Before we begin with the questions, it would be helpful to know generally the total number of combinations of 5 card hands from the 52 card deck. This should not challenge you too much:

$$C(52, 5) = 2,598,960 \text{ possible hands.}$$

Remember, the easiest way to solve these problems is to use the standard counting methods to find all the specified combinations, and then divide by the total number of possibilities. This will give you the probability.

(a) What are the odds of getting a *royal flush*? (A sequence of A, K, Q, J, 10, all of the same suit.)

This question is on your coursework. Work out for yourself!

(b) What are the odds of a *straight flush*? (A sequence of 5 cards, all of the same suit, but not a royal flush.)

Once again, it is quite simple to work out the possible number of sequences. If the lowest number in the sequence is 10, then the resulting straight will be a royal flush, which we don't want in this case. This leaves 9 other numbers that can start our straight. Multiply this by the four possible suits, and there are 36 possible straight flushes.

The odds of a straight flush are 1 in $72,193$ (when rounded.)

(c) What are the odds of *four-of-a-kind*? (Four cards of the same kind, e.g., four aces.)

This question is on your coursework. Work out for yourself!

(d) What are the odds of a *full house*? (A hand with cards of one kind, and two or another.)

You can actually use the same method as with the previous part to determine the odds for a full house. The numbers change slightly, of course: $C(13, 1) \cdot C(4, 3) \cdot C(12, 1) \cdot C(4, 2) = 3,744$.

The odds for full house are 1 in 694.

(e) What are the odds of a *flush*? (A hand that contains any 5 cards but are all of the same suit.)

This example has been omitted as it relies on knowing the number of royal flushes.

(f) What are the odds of a *straight*? (Any sequence of 5 cards, regardless of their suit.)

This question is on your coursework. Work out for yourself!

(g) What are the odds of a *three-of-a-kind*? (A hand with three cards of the same kind.)

Think of the workings for a full house. However, in this instance, instead of having a pair with the three-of-a-kind, you can have any two cards. $C(13,1) \cdot C(4,3) \cdot C(12,2) \cdot 4^2 = 54,912$.

The odds of a three-of-a-kind are 1 in 47.

(h) What are the odds of two pairs?

To begin, we choose two different numbers, $C(13,2)$.To match one of those numbers, there are $C(4,2)$ ways from the four suits. And then the same again for the second card. This gives us our two pairs. To pick the fifth card, there are 48 remaining in the deck. However, we don't want either of the remaining numbers that we already have, so discard those 4 cards, which leaves 44 cards to choose from. This gives: $C(13,2) \cdot C(4,2) \cdot C(4,2) \cdot 44 = 123,552$ combinations.

The odds of two pairs are 1 in 21.

(i) What are the odds of a *pair*?

Choose a number and then find how many ways to select it twice from the four suits: $C(13,1) \cdot C(4,2)$. Now pick any three from the remaining 12. They can be of any suit, so multiply accordingly. This means the total number of pairs is: $C(13,1) \cdot C(4,2) \cdot C(12,3) \cdot 4^3 = 1,098,240$.

The odds of a pair are 1 in 2 (when rounded.)

(j) What are the odds of a *high card*? (5 cards, all distinct, not forming a straight and not all of the same suit.)

Now, when you haven't got any of the above hands, you can still play with your highest card. Firstly, choosing 5 numbers is $C(13,5)$, but don't forget to remove the 10 straights. Now, each of the 5 cards can be of any suit, but ensuring that it can't be the 4 occasions that there are all of the same suit, this results in: $(C(13,5) - 10)\,(4^5 - 4) = 1,277 \cdot 1,020 = 1,302,540$ distinct hands.

The odds of a high card are 1 in 2.