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Probability Theory – Part II

Conditional Probability

So far, we have largely being focusing on calculating the probability of a single event. However, we will now look at how it is possible to find the probability of event *B*, given that event *A* has already occurred.

It is important that you are familiar with the appropriate notation:

$P(A \cap B)$ *the probability that both event A and event B occur*

$P(A \mid B)$ *the probability of event A occurring, given that event B has already happened*

To calculate the conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Let us begin is a typical top example of using this technique.

Example:

(1) A box contains 4 red balls and 3 blue balls. Two balls are taken from the box, and are not replaced. Let event *A* be "red drawn first" and *B* is "blue drawn second".

a) What is $P(A \cap B)$?

It should be obvious that $P(A) = \frac{4}{7}$. And if event A has occurred, that means there are 3 red and 3 blue left (a total of 6 balls). Therefore,

$$P(A \cap B) = \tfrac{4}{7} \times \tfrac{3}{6} = \tfrac{2}{7}$$

b) What is $P(B \mid A)$?

We can use the conditional probability equation above and substitute in the values we know.

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{7}}{\frac{4}{7}} = \frac{1}{2}$$

c) What is $P(A \mid B)$?

Before we can work this out, we must find *P(B)*, which is still unknown. There are two ways to for event *B* to happen. We could draw a red first and then a blue, or a blue first, and then another blue.

$$\begin{aligned} P(B) &= [red\ \ first \cap blue\ second] \cup [blue\ \ first \cap blue\ second] \\ &= P(red\ \ first \cap blue\ second) + P(blue\ \ first \cap blue\ second) \\ &= P(blue\ second \mid red\ \ first)P(red\ \ first) \\ &\quad + P(blue\ second \cap blue\ \ first)P(blue\ \ first) \\ &= \frac{1}{2}\cdot\frac{4}{7}+\frac{2}{6}\cdot\frac{3}{7} \\ &= \frac{3}{7}. \end{aligned}$$

This result can be plugged straight into the usually equation:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{7}}{\frac{3}{7}} = \frac{2}{3}$$

(2) On average when ill, 50% of people have a headache, 10% have a fever, and 90% of people with a fever also have a headache. Given that I have a headache, what is the probability I also have a fever? Let event H be “have headache” and F is “have fever”.

From the question, pick out the important facts:

$$P(F) = \tfrac{1}{10}$$
$$P(H) = \tfrac{1}{2}$$
$$P(H \mid F) = \tfrac{9}{10}$$

We want $P(F \mid H)$.

$$P(F \mid H) = \frac{P(H \cap F)}{P(H)}$$

We already know what *P(H)* is, so we need a way to find $P(H \cap F)$. Remember, we do know what $P(H \mid F)$ equals:

$$P(H \mid F) = \frac{P(F \cap H)}{P(F)} = \frac{P(F \cap H)}{\frac{1}{10}} = \frac{9}{10}$$

It should be obvious that $P(H \cap F) = P(F \cap H)$, which we can find from the above equation, which gives $P(H \cap F) = P(F \cap H) = \frac{9}{100}$. So, with all the necessary unknowns now found, it is trivial to substitute into the relevant equation to find the answer:

$$P(F \mid H) = \frac{P(H \cap F)}{P(H)} = \frac{\frac{9}{100}}{\frac{1}{2}} = \frac{9}{50}$$

(3) In a certain school, 25% students failed maths, 15% failed chemistry, and 10% failed both maths and chemistry. Let event *M* be "failed maths" and *C* is "failed chemistry".

a) If a student failed chemistry, what is the probability they also failed maths?

Using the general conditional probability equation, we can find this easily:

$$P(M \mid C) = \frac{P(M \cap C)}{P(C)} = \frac{\frac{1}{10}}{\frac{3}{20}} = \frac{2}{3}$$ (NB 15% = $\frac{3}{20}$)

b) If a student failed maths, what is the probability they also failed chemistry?

$$P(C \mid M) = \frac{P(C \cap M)}{P(M)} = \frac{\frac{1}{10}}{\frac{1}{4}} = \frac{2}{5}$$

c) What is the probability that a student failed maths or failed chemistry?

From your notes, you should be familiar with the following equation:

$$\begin{aligned} P(M \cup C) &= P(M) + P(C) - P(M \cap C) \\ &= \tfrac{1}{4} + \tfrac{3}{20} - \tfrac{1}{10} = \tfrac{3}{10} \end{aligned}$$

d) What is the probability that a student failed neither?

This is quite simple. We know the probability for failing either, so this can simply be subtracted from 1:

$$1 - P(either) = 1 - \tfrac{3}{10} = \tfrac{7}{10}$$

Independence

Determining whether one or more events are independent of each other is a valuable property to know in probability. It should be obvious for example, that when tossing a die, getting a head first, and a head second, are not related, since the coin has no memory, and so should act completely randomly. However, as we saw in the last example, drawing a blue ball second was very much dependent on the previous event.

There are two methods to determine this:

1. If events A and B are independent, then $P(B \mid A) = P(B)$ since event A has no influence on event B.

2. Given the above result, $P(B \mid A) = P(B) = \frac{P(A \cap B)}{P(A)}$. With a little rearrangement, you get:

$$P(A \cap B) = P(A)P(B)$$

This will only be equal when the events are independent.

Example:

(1) Consider the following data on car ownership of pink cars and poodles in a community. Let event *A* be "owning a pink car" and event *B* be "owning a poodle". Are *A* and *B* independent?

	Own Poodles	Don't Own Poodles	*Totals*
Own pink cars	250	450	*700*
Don't own pink car	1,250	18,350	*19600*
Totals	**1,500**	**18,800**	***20,300***

From the data, we can quickly obtain:

$P(A) = \frac{700}{20300}$ $P(B) = \frac{1500}{20300}$ $P(A \cap B) = \frac{250}{20300}$

It should be obvious if you are familiar with multiplication of fractions that if the probabilities of *A* and *B* are multiplied, you will get a significantly smaller number compared to $P(A \cap B)$.

I shall leave it up to you to check with your calculators if you wish, but take it from me that:

$$P(A \cap B) \neq P(A)P(B)$$

Therefore, events *A* and *B* are not independent.

(2) The probability of it raining to today is 0.3. Also, 40% of all rainy days are followed by rainy days and 20% of all days without rain are followed by rainy days. If event A is "rain today", and event be is "rain tomorrow". Find:

a) $P(B \mid A)$

This is not actually at all taxing as the answer is given in the question. The chances of it raining tomorrow, given that it has rained today is 0.4 (40%).

b) $P(A \cap B)$, i.e., the probability of it raining on both days.

Simple manipulation will yield:

$$P(A \cap B) = P(A \mid B)P(B)$$

The probabilities on the right-hand side are already known.

$$P(A \cap B) = P(A \mid B)P(B) = 0.4 \times 0.3 = 0.12$$

c) Are events *A* and *B* independent?

From the information in the question, it is possible to calculate that $P(B) = 0.26$. I'll leave it as an exercise to the reader to work out how that result is found (it will be good practise!) Providing you trust my workings, it is now fairly trivial to perform a test for independence:

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ &= 0.3 \times 0.26 = 0.078 \end{aligned}$$

However, we know from part *b)* that $P(A \cap B) = 0.12$. Therefore, the events are not independent.

(3) Consider the following events for a family with children. Let event *A* be "children of both sexes", and event *B* is "at most one boy".

a) Show that *A* and *B* are independent if the family has 3 children.

Let us first find the appropriate sample space:

$$\begin{aligned} S &= \{\text{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg}\} \\ A &= \{\text{bbg, bgb, bgg, gbb, gbg, ggb}\} \\ B &= \{\text{bgg, gbg, ggb, ggg}\} \\ A \cap B &= \{\text{bgg, gbg, ggb}\} \end{aligned}$$

With this information, we can easily find the respective probabilities:

$$P(A) = \tfrac{6}{8} = \tfrac{3}{4}$$
$$P(B) = \tfrac{4}{8} = \tfrac{1}{2}$$
$$P(A \cap B) = \tfrac{3}{8}$$

If *A* and *B* are independent, then $P(A)P(B) = P(A \cap B)$:

$$P(A)P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$$ therefore they are independent.

b) What about if the family has 2 children?

Continuing in a similar fashion to the previous question:

$$
\begin{aligned}
S &= \{bb, bg, gb, gg\} \\
A &= \{bg, gb\} \\
B &= \{bg, gb, gg\} \\
A \cap B &= \{bg, gb\}
\end{aligned}
$$

With this information, we can easily find the respective probabilities:

$$
\begin{aligned}
&P(A) = \tfrac{2}{4} = \tfrac{1}{2} \\
&P(B) = \tfrac{3}{4} \\
&P(A \cap B) = \tfrac{2}{4} = \tfrac{1}{2}
\end{aligned}
$$

If *A* and *B* are independent, then $P(A)P(B) = P(A \cap B)$:

$P(A)P(B) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8} \neq P(A \cap B)$ therefore they are not independent.